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Completeness Conditions for Boundary Operators in 2D Conformal Field Theory

G. Pradisi, A. Sagnotti and Ya.S. Stanev [1]

Dipartimento di Fisica
Università di Roma "Tor Vergata"
I.N.F.N. - Sezione di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1 00133 Roma ITALY

Abstract

In non-diagonal conformal models, the boundary fields are not directly related to the bulk spectrum. We illustrate some of their features by completing previous work of Lewellen on sewing constraints for conformal theories in the presence of boundaries. As a result, we include additional open sectors in the descendants of D_{odd} $SU(2)$ WZW models. A new phenomenon emerges, the appearance of multiplicities and fixed-point ambiguities in the boundary algebra not inherited from the closed sector. We conclude by deriving a set of polynomial equations, similar to those satisfied by the fusion-rule coefficients N_{ij}^{k}, for a new tensor A_{ab}^{i} that determines the open spectrum.

[1] I.N.F.N. Fellow, on Leave from Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, BG-1784 Sofia, BULGARIA.

Introduction

Cardy showed [1] that in diagonal rational conformal models the admissible types of boundaries are in one-to-one correspondence with the bulk fields. This observation has provided a convenient setting [2, 3] for the proposal [4] of associating open descendants to closed oriented models, since different types of boundaries correspond to different types of Chan-Paton groups [5]. The descendants [6, 7] of $SU(2)$ WZW models [8] revealed new aspects of the problem, most notably the possibility of different Klein-bottle projections of the bulk spectrum, fully determined by (a proper extension of) the crosscap constraint of ref. [9].

The descendants of non-diagonal models are harder to deal with, since the simple correspondence between bulk and boundary sectors is lost in this case[2]. In ref. [7] we solved the diophantine equations for the non-diagonal $SU(2)$ WZW models, introducing $\rho + 3$ charge sectors in the D_{odd} models with level $k = 4\rho + 2$. This result was obtained under the seemingly plausible assumption that all multiplicities in the annulus amplitude be inherited from the closed spectrum, as for the D_{even} models discussed in ref. [6]. The final result, however, is rather puzzling, since the arguments of ref. [2] would suggest a number of allowed boundary sectors equal to the number of characters paired with their conjugates by the bulk GSO projection, namely $2\rho + 3$.

In this letter we reconsider the issue, by first completing and partly correcting some previous interesting work of Lewellen [10] on sewing constraints for conformal models in the presence of boundaries. These constraints lead to a set of equations, sufficient to determine the vacuum channel of the annulus amplitude, that are solved explicitly for the D_{odd} models. Moreover, we prove that *the general solution to the problem of classifying boundary conditions is given in terms of an integer-valued tensor A^i_{ab} satisfying a set of polynomial equations.* These may be regarded as completeness conditions for the allowed boundaries.

For the boundary algebra, A^i_{ab} plays a role similar to that played by the fusion-rule

[2]Rather unconventionally, by a diagonal model we always mean one built with the charge-conjugation matrix C of the fusion algebra.

coefficients N_{ij}^{k} for the bulk algebra. The additional charges missed in ref. [7] reveal a rather amusing and unexpected feature: *the boundary algebra of non-diagonal models can be extended even when the bulk algebra can not*, since the boundary states generically correspond to (normalized) combinations of those allowed in the diagonal case. As a result, the annulus amplitude may contain some multiplicities that draw their origin solely from the open sector.

Aside from the application to open-string models, the construction of open descendants has some interest for Statistical Mechanics [11], as well as for the emerging picture of non-perturbative string dynamics, where boundaries play an essential role [12, 13], since the available choices of conformally invariant boundary conditions determine the possible types of (generalized) D branes.

Sewing Constraints for the Annulus Amplitude

Sewing constraints for conformal models in the presence of boundaries were first discussed by Lewellen [10], but the original derivation contains some errors, and some of his final results need modifications. In the spirit of ref. [4], let us assume to have solved the "parent" theory, so as to know its modular matrices T and S, the braid matrices B and the duality matrices F [14], the fusion-rule coefficients N_{ij}^{k} and the bulk OPE coefficients $C_{(i\bar{i})(j\bar{j})}^{(k\bar{k})}$. For a diagonal model, the open descendants are determined to a large extent by the construction of refs. [2], based on Cardy's ansatz [1] for the annulus spectrum. Multiple Klein-bottle projections of the bulk spectrum were discussed in refs. [6, 7], where (a proper extension of) the "crosscap" constraint of ref. [9] was used to determine them for all $SU(2)$ WZW models[3]. For non-diagonal models, one does not have so far an equivalent recipe, and the main purpose of this letter is to set the stage for this more general case.

Denoting the "bulk fields" of the theory by $\phi_{i,\bar{i}}$ and the "boundary fields" by ${\psi_i}^{ab}$, one

[3]In these models, the open spectrum contains a simple current that connects pairs of different charge assignments.

has the usual bulk OPE, as well as the boundary OPE

$$\psi_i{}^{ab}\ \psi_j{}^{bc} \sim \sum_l\ C^{abc}_{ijl}\ \psi^{ac}_l \qquad . \tag{1}$$

Additional data of the descendant models are the normalizations of two-point functions, $\alpha_i{}^{ab}$, defined by

$$< \psi_i{}^{ab}(x_1)\ \psi_i{}^{ba}(x_2) > = \frac{\alpha_i{}^{ab}}{(x_{12})^{2\Delta_i}} \quad , \tag{2}$$

where for $SU(2)$ WZW models

$$\alpha_i{}^{ab}\ =\ \alpha_i{}^{ba}\ (-1)^{2I_i} \quad , \tag{3}$$

with I_i the isospin of ψ_i. While restricting our attention to $SU(2)$ WZW models, we would like to point out that all our formulas may be turned into corresponding ones for minimal models, provided all isospin-dependent factors are set to one. Making use of eq. (1) in the three-point functions of boundary operators $< \psi_i{}^{ab}\psi_j{}^{bc}\psi_l{}^{ca} >$ and $< \psi_j{}^{bc}\psi_l{}^{ca}\psi_i{}^{ab} >$ leads to

$$C^{abc}_{ijl}\ \alpha_l{}^{ac}\ =\ C^{bca}_{jli}\ \alpha_i{}^{ab} \qquad \text{and} \qquad C^{bca}_{jli}\ \alpha_i{}^{ba}\ =\ C^{cab}_{lij}\ \alpha_j{}^{bc} \quad . \tag{4}$$

These two relations may be connected using eq. (3), and one finally obtains

$$C^{abc}_{ijl}\ \alpha_l{}^{ac}\ =\ (-1)^{2I_i}\ C^{cab}_{lij}\ \alpha_j{}^{bc} \qquad . \tag{5}$$

Moreover, the proper behavior of the identity requires that

$$C^{abb}_{i\mathbf{1}i}\ =\ 1 \qquad \text{and} \qquad < \mathbf{1}^{aa} > =\ \alpha^{aa}_{\mathbf{1}} \qquad , \tag{6}$$

while all other one-point functions of boundary fields vanish.

One may then proceed to consider amplitudes $< \psi^{ab}_i\ \psi^{bc}_j\ \psi^{cd}_k\ \psi^{da}_l >$ for four boundary operators. Demanding that their s and u-channel expansions coincide ("planar duality" for open four-point amplitudes) yields

$$\sum_p\ C^{abc}_{ijp}\ C^{cda}_{klp}\ \alpha^{ac}_p\ S_p(i,j,k,l)\ =\ \sum_q\ C^{bcd}_{jkq}\ C^{dab}_{liq}\ \alpha^{bd}_q\ U_q(i,j,k,l) \quad , \tag{7}$$

and relating the u-channel blocks to the s-channel ones by the fusion matrix F

$$U_q\ =\ \sum_p\ F_{qp}\ S_p \tag{8}$$

turns eq. (7) into

$$C^{abc}_{ijp}\, C^{cda}_{klp}\, \alpha^{ac}_{p} \;=\; \sum_q \; C^{bcd}_{jkq}\, C^{dab}_{liq}\, \alpha^{bd}_{q}\; F_{qp}(i,j,k,l) \quad , \tag{9}$$

a quadratic constraint for the boundary OPE coefficients C^{abc}_{ijk} and the normalizations α^{ab}_{i} of the two-point functions of boundary fields.

The last crucial ingredient of the construction, introduced in refs. [15, 10], is the OPE for bulk fields in front of a boundary. This corresponds to a familiar intuitive picture: when a bulk field approaches a boundary, the result should be expressible solely in terms of boundary fields. Thus,

$$\phi_{i,\bar{i}} \;\sim\; \sum_j \; C^{a}_{(i,\bar{i})j}\; {\psi_j}^{aa} \quad , \tag{10}$$

where the proper behavior of the identity requires that

$$C^{a}_{(\mathbf{1},\mathbf{1})\mathbf{1}} \;=\; 1 \qquad . \tag{11}$$

One may then proceed to consider amplitudes for one bulk field and two boundary fields. There are two ways of computing $<\phi_{(i,\bar{i})}\; \psi^{ba}_{j}\; \psi^{ab}_{k}>$, according to which portion of the boundary the bulk field faces, and the resulting condition is

$$\sum_l \; C^{b}_{(i,\bar{i})l}\; C^{bba}_{ljk}\; \alpha^{ba}_{k}\; S_l(i,\bar{i},j,k) \;=\; \sum_n \; C^{a}_{(i,\bar{i})n}\; C^{baa}_{jnk}\; \alpha^{ba}_{k}\; U_n(j,i,\bar{i},k) \quad . \tag{12}$$

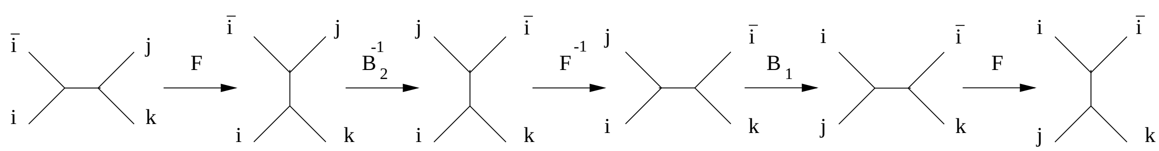

Figure 1 - Analytic continuation of $<\phi\;\psi\;\psi>$

Finding the proper relation between the blocks U_n and S_l involves some delicate analytic continuations, and we disagree with the final result of ref. [10]. In order to derive this sewing constraint, we shall reduce the relevant transformation to a sequence of elementary moves, as in fig. 1. These comprise two basic operations, the braidings B_i of pairs of nearby operators and the fusion F [14], and the result reads

$$\begin{aligned} U_n(j,i,\bar{i},k) \quad = \quad & \sum_{m,r,s,p,l} F_{nm}(j,i,\bar{i},k)\; (B_1)_{mr}(i,j,\bar{i},k)\; F^{-1}_{rs}(i,j,\bar{i},k) \\ & (B_2)^{-1}_{sp}(i,\bar{i},j,k)\; F_{pl}(i,\bar{i},j,k)\; S_l(i,\bar{i},j,k) \qquad . \end{aligned} \tag{13}$$

Substituting in eq. (12) and recalling that in this case the braid matrices B_i are diagonal [16] yields the final form of the constraint,

$$C^{b}_{(i,\bar{i})l}\ C^{bab}_{jkl}\ {\alpha_l}^{bb} \quad = \quad \sum_{m,n,p} C^{a}_{(i,\bar{i})n}\ C^{aba}_{kjn}\ {\alpha_n}^{aa}(-1)^{(I_i-I_{\bar{i}}+2I_j+I_p-I_m)}\ e^{-i\pi(\Delta_i-\Delta_{\bar{i}}-\Delta_m+\Delta_p)}$$
$$F_{nm}(j,i,\bar{i},k)\ {F^{-1}}_{mp}(i,j,\bar{i},k)\ F_{pl}(i,\bar{i},j,k) \qquad . \tag{14}$$

Similar considerations apply to the last constraint of ref. [10]. This results from the comparison between two different definitions of three-point amplitudes for two bulk fields and one boundary field, $< \phi_{(i,\bar{i})}\ \phi_{(j,\bar{j})}\ {\psi_k}^{aa} >$. These are effectively chiral five-point amplitudes, and are thus more complicated than the previous ones. In this case, the first definition uses the bulk OPE, while the second definition uses a pair of bulk-boundary OPE's. Demanding that the two resulting expressions coincide gives

$$\sum_{p,q} C^{(p,\bar{q})}_{(j,\bar{i})(i,\bar{j})}\ C^{a}_{(p,\bar{q})k}\ \alpha^{aa}_k\ Y_{p\bar{q}}(j,i,\bar{i},\bar{j},k) \ = \ \sum_{p,q} C^{a}_{(i,\bar{i})p}\ C^{a}_{(j,\bar{j})q}\ C^{aaa}_{pqk}\ \alpha^{aa}_k\ X_{pq}(i,\bar{i},j,\bar{j},k) \quad . \tag{15}$$

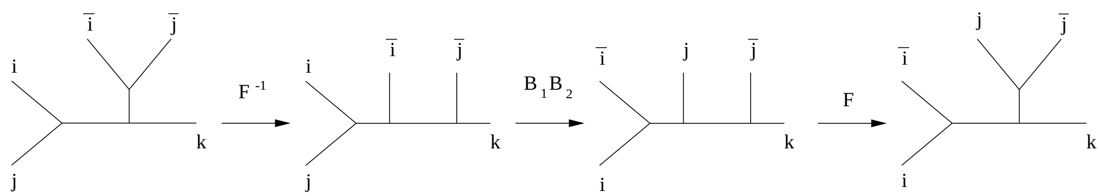

Figure 2 - Analytic contiuation of $<\phi\ \phi\ \psi>$

Again, relating the two expressions requires a careful analytic continuation, that may be reduced to the sequence of elementary moves displayed in fig. 2. The resulting constraint, present only if α^{aa}_k does not vanish, reads

$$C^{(p,\bar{q})}_{(j,\bar{i})(i,\bar{j})}\ C^{a}_{(p,\bar{q})k} \quad = \quad \sum_{r,s,t} (-1)^{(I_j-I_t+I_r)}\ e^{-i\pi(\Delta_j-\Delta_t+\Delta_r)}\ C^{a}_{(i,\bar{i})r}\ C^{a}_{(j,\bar{j})s}\ C^{aaa}_{rsk}$$
$$F_{st}(r,j,\bar{j},k)\ F_{rp}(j,i,\bar{i},t)\ F^{-1}_{t\bar{q}}(p,\bar{i},\bar{j},k) \quad . \tag{16}$$

For $SU(2)$ WZW models the bulk OPE coefficients satisfy

$$C^{(p,\bar{q})}_{(j,\bar{i})(i,\bar{j})} \ = \ (-1)^{I_i+I_j-I_p}\ C^{(p,\bar{q})}_{(i,\bar{i})(j,\bar{j})} \qquad , \tag{17}$$

and are normalized in a different fashion with respect to ref. [7], so that now

$$C^{(p,q)}_{(i,k)(j,l)} \ = \ \epsilon^{(p,q)}_{(i,j)(k,l)} \sqrt{\frac{C_{ijp}C_{klq}}{C_{pp\mathbf{1}}C_{qq\mathbf{1}}}} \quad , \tag{18}$$

since this choice simplifies our final expressions. The rather complicated expressions for F and C_{ijk} depend on normalization choices. We follow ref. [16], while some of the relevant formulas may also be found in ref. [7]. Moreover, the ϵ's are signs present only for the non-diagonal models. In the next section we shall specify them explicitly for the cases of interest.

We have thus completed the derivation of the sewing constraints. The resulting relations (and their solutions) differ from those obtained in ref. [10] even for the simplest case of the Ising model. In the $SU(2)$ WZW models a subset of these constraints, sufficient to determine the annulus vacuum amplitude, decouples. In order to elucidate this point, let us confine our attention to the amplitude $< \phi_{(i,\bar{\imath})} \; \phi_{(j,\bar{\jmath})} \; \mathbf{1}^{aa} >$, whereby eq. (16) becomes

$$\begin{aligned} C^{(q,\bar{q})}_{(j,\bar{\imath})(i,\bar{\jmath})} \; C^{a}_{(q,\bar{q})\mathbf{1}} \; \alpha^{aa}_{\mathbf{1}} \quad &= \quad \sum_p \, (-1)^{(I_j - I_{\bar{\jmath}} + I_p)} \; e^{-i\pi(\Delta_j - \Delta_{\bar{\jmath}} + \Delta_p)} \\ & \qquad \alpha^{aa}_p \; C^{a}_{(i,\bar{\imath})p} \; C^{a}_{(j,\bar{\jmath})p} \; F_{pq}(j,i,\bar{\imath},\bar{\jmath}) \quad . \end{aligned} \tag{19}$$

Multiplying by $F^{-1}_{qr}(j,i,\bar{\imath},\bar{\jmath})$, singling out the identity (*i.e.* choosing $r = \mathbf{1}$), defining

$$B^{a}_{(i,\bar{\imath})} = C^{a}_{(i,\bar{\imath})\mathbf{1}} \tag{20}$$

and using the explicit expressions for fusion matrices and structure constants of ref. [16] yields a set of relations involving *only* the B's, the basic ingredients of the annulus vacuum channel. Since the $\alpha^{aa}_{\mathbf{1}}$ never vanish, one obtains the simple constraint

$$B^a_i \; B^a_j \;\; = \;\; \sum_l \; \epsilon^l_{ij} \; N^l_{ij} \; B^a_l \qquad , \tag{21}$$

where we have expressed the restriction of the sum to all terms in the "fusion range" of i and j via the fusion-rule coefficients N^k_{ij} and we have simplified the notation, replacing every pair of (coincident) indices with a single one.

Once the solution to eq. (21) has been found, the vacuum-channel annulus amplitude is

$$\tilde{A} \;\; = \;\; \frac{1}{2} \sum_i \frac{\chi_i}{[2I+1]} \left(\sum_a B^a_i \; n^a \; \alpha^{aa}_{\mathbf{1}} \right)^2 \quad , \tag{22}$$

since we have normalized the two-point functions of the bulk fields to their quantum dimensions $[2I+1]$ while, in general, the direct channel annulus amplitude is of the form

$$A \;\; = \frac{1}{2} \; \sum_{abi} A^i_{ab} \; n^a \; n^b \; \chi_i \qquad . \tag{23}$$

In open-string theories, the non-negative integers A^i_{ab} encode the properties of the open sector. In rational conformal models, they determine the set of conformally invariant boundary conditions or, equivalently, they count the boundary fields ψ_i^{ab}.

Although we have derived eq. (21) in the context of $SU(2)$ WZW models, a generalization holds in all cases. For instance, in toroidal models one can show that the most general modification of the vacuum-channel coefficients involves multiplicative phases. These are just the Wilson lines of ref. [3] that implement in open-string models the construction of ref. [17].

Application to the D_{odd} $SU(2)$ WZW Models

It is instructive to apply the results of the preceding section to the D_5 model, the simplest non-diagonal $SU(2)$ WZW model. This will allow us to reconsider the construction of ref. [7], that actually turns out to be incomplete. In that paper we found by brute force only four boundary sectors, a somewhat surprising result in view of the conventional wisdom, that leads one to expect five sectors, as many as the bulk fields allowed in the annulus vacuum channel. Indeed, the B_i^a determine the number of independent combinations of the n's, and thus the independent charge sectors of the model, but they also determine the one-point functions of the bulk fields in front of a boundary, via their products with the α's. Since these one-point functions are essentially chiral two-point functions on the sphere, non-vanishing results obtain *only* for the fields that the GSO projection of the bulk spectrum mixes with their charge conjugates.

The spectrum of the D_5 model in the ADE classification [18] is described by

$$T \;=\; |\chi_1|^2 \;+\; |\chi_3|^2 \;+\; |\chi_5|^2 \;+\; |\chi_7|^2 \;+\; |\chi_4|^2 \;+ \chi_2\bar{\chi}_6 + \chi_6\bar{\chi}_2 \quad , \tag{24}$$

where the subscript of χ_i is related to its isospin I by $i = 2I+1$. Since all these characters are self-conjugate, one would indeed expect five charge sectors, whereas in ref. [7] we could only identify four of them.

One may apply eq. (21) to this case, noting that the only difference between the two systems of quadratic equations for the A_6 and D_5 models lies in the signs ϵ^l_{ij}, all equal to

one in the diagonal model, and the two subsystems for the integer-isospin coefficients are identical. For the diagonal A_6 model, the system has a total of seven distinct solutions. Each complete choice of coefficients determines, according to eq. (22), the contribution of one type of Chan-Paton charge to the vacuum amplitude. Strictly speaking, this would also require the α's, that in the diagonal model can be determined as in ref. [1]. More generally, one may determine them completely turning the annulus amplitude to the direct channel by a modular S transformation and requiring that the coefficients be (half)integer. For the diagonal model one thus recovers the seven types of charges of ref. [6]. The resulting correspondence between charge types and open-string sectors is displayed in the first column of the table.

B coefficients for the A_6 and D_5 models								
Op's	B_1	B_3	B_5	B_7	$B_4^{(A_6)}$	B_2	B_6	$B_4^{(D_5)}$
$(\frac{1}{2},\frac{5}{2})$	1	1	-1	-1	0	$\pm\sqrt{2}$	$\mp\sqrt{2}$	0
$(\frac{3}{2})$	1	-1	1	-1	0	0	0	± 2
$(0,3)$	1	$1+\sqrt{2}$	$1+\sqrt{2}$	1	$\pm\sqrt{2(2+\sqrt{2})}$	$\pm\sqrt{2+\sqrt{2}}$	$\pm\sqrt{2+\sqrt{2}}$	0
$(1,2)$	1	$1-\sqrt{2}$	$1-\sqrt{2}$	1	$\mp\sqrt{2(2+\sqrt{2})}$	$\pm\sqrt{2-\sqrt{2}}$	$\pm\sqrt{2-\sqrt{2}}$	0

Extending this analysis to the D_5 model is particularly rewarding. In this case both B_2 and B_6 vanish for the general argument discussed above, while the new equations for B_4,

$$\begin{aligned} B_4\ B_{2I+1} &= (-1)^I B_4 \quad , \\ B_4\ B_4 &= B_1 - B_3 + B_5 - B_7 \quad , \end{aligned} \tag{25}$$

involve some additional signs introduced by the ϵ's. B_4 thus vanishes, unless B_{2I+1} is precisely $(-1)^I$. This occurs in the third row of the table, but in this case there are two solutions, $B_4 = \pm 2$, as displayed in the last column. A closer inspection of the table reveals the correspondence between the boundary states of the two models, determined by the condition that both B_2 and B_6 vanish in the non-diagonal case. Some of the new boundary states are created from the vacuum by the following linear combinations of the

boundary operators of the diagonal model:

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{2}}(\psi_2^{2,1} + \psi_6^{6,1}) \\ \xi_2 &= \frac{1}{\sqrt{2}}(\psi_1^{1,1} + \psi_7^{7,1}) \\ \xi_5 &= \frac{1}{\sqrt{2}}(\psi_3^{3,1} + \psi_5^{5,1})\end{aligned} \tag{26}$$

where, again, all labels on the *r.h.s.* correspond to $2I + 1$. The correspondence, however, is only partial, since there are now *two* boundary sectors ξ_3 and ξ_4 corresponding to the middle field ψ_4 ! A fixed-point ambiguity, not present in the spectrum of bulk fields, has emerged in the set of boundary fields. This is not the only surprise, since we did get these charges in ref. [7]. What we missed there was the charge sector corresponding to ξ_5, since we allowed no multiplicities in the boundary fusion algebra, and thus in the direct-channel annulus amplitude. Multiplicities are indeed present, as may be foreseen from eq. (26), since the fusion of ξ_5 is

$$[\xi_5] \ \times \ [\xi_5] \ = \ [\xi_2] \ + \ 2\ [\xi_5] \quad . \tag{27}$$

Thus, as compared to the diagonal case, the algebra of boundary operators is an extended algebra. This is rather amusing, since the simple current in this case has dimension 3/2, and therefore does not extend the bulk algebra. As a result, the complete open sector of the model with "real" charges is described by

$$\begin{aligned} A \ = \ \frac{1}{2} \Big(& \chi_1(l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2) \ + \ (\chi_2 + \chi_6)(2l_1l_2 + 2l_1l_5 + 2l_3l_5 + 2l_4l_5) \ + \\ & \chi_3(l_1^2 + 2l_1l_3 + 2l_1l_4 + 2l_3l_4 + 2l_2l_5 + 2l_5^2) \ + \\ & \chi_4(4l_1l_5 + 2l_2l_3 + 2l_3l_5 + 2l_2l_4 + 2l_4l_5) \ + \\ & \chi_5(l_1^2 + l_3^2 + l_4^2 + 2l_5^2 + 2l_1l_3 + 2l_1l_4 + 2l_2l_5) \ + \ \chi_7(l_1^2 + l_2^2 + l_5^2 + 2l_3l_4)\Big) \quad , \end{aligned} \tag{28}$$

and

$$\begin{aligned} M \ = \ \pm \frac{1}{2} \Big(& \hat{\chi}_1(l_1 - l_2 + l_3 + l_4 - l_5) \ + \ \hat{\chi}_3(-l_1 + 2l_5) \ + \\ & \hat{\chi}_5(l_1 + l_3 + l_4) \ + \ \hat{\chi}_7(l_1 + l_2 + l_5)\Big) \end{aligned} \tag{29}$$

where the labels of the charges correspond to those of the ξ's. Indeed, the new charge l_5 has multiplicities both in the annulus and in the Möbius amplitude. The model with complex charges involves similar modifications.

These results may be extended to all D_{odd} models, thus allowing for a total of $2\rho+3$ charges. The resulting assignments may be described rather neatly in terms of an auxiliary diagonal model so that, in the notation of eq. (37) in ref. [7], the complete embedding for the case of real charges is

$$n_a \;=\; n_a^+ \;+\; n_a^- \;+\; \frac{i}{2\sqrt{\rho+1}}\; O_a (-1)^{\frac{a-1}{2}}\; (l_{\rho+2} - l_{\rho+3}) \qquad (\mathrm{a} \;= 1,\; \ldots\, , k+2) \quad , \tag{30}$$

where O_a denotes the projector on odd a and the $n^{\pm}$ satisfy the relations

$$\begin{aligned}
& n^{\pm}_{\frac{k+2}{2}+b} \;=\; n^{\pm}_{\frac{k+2}{2}-b} \quad , \qquad n^{\pm}_{\frac{k+2}{4}+b} \;=\; \pm\, n^{\pm}_{\frac{k+2}{4}-b} \qquad (b \geq 1) \quad , \\
& n^-_b \;=\; -\,\frac{l_{b+1}}{2} \quad , \qquad n^+_b \;=\; \frac{l_{\rho+3+b}}{2} \quad , \qquad (1 \leq b \leq \rho) \quad , \\
& n^-_{\frac{k+2}{2}} \;+\; n^+_{\frac{k+2}{2}} \;=\; l_1 \quad , \qquad n^-_{\frac{k+2}{4}} \;+\; n^+_{\frac{k+2}{4}} \;=\; \frac{1}{2}\,(l_{\rho+2} + l_{\rho+3}) \quad .
\end{aligned} \tag{31}$$

Completeness Conditions for Boundary Operators

A closer inspection of eq. (28) reveals a very interesting property. Namely, it may be verified that the non-negative integers A^i_{ab} defined in eq. (23) satisfy two sets of polynomial equations involving also the fusion-rule coefficients N^k_{ij},

$$\sum_b \; A^{i\,b}_{\;\;a}\; A^j_{bc} \;\;=\;\; \sum_k N^{ij}_k\; A^k_{ac} \qquad , \tag{32}$$

$$\sum_i \; A_{iab}\; A^i_{cd} \;\;=\;\; \sum_i \; A_{iac}\; A^i_{bd} \qquad , \tag{33}$$

while omitting the l_5 terms would violate eq. (32). Upper and lower boundary indices are to be distinguished whenever complex charges (corresponding to oriented boundaries) are present. The matrix $(A_1)_{ab} = (A_1)^{ab}$ is a metric for the boundary indices, since it follows from eq. (32) that $\sum_b A_{iab}\;\, {A_1}^{bc} = A_{i\;a}^{\;\,c}$, while $(A_1)^b_a = \delta^b_a$. In diagonal models, where A coincides with N, these equations reduce to the Verlinde algebra.

One can prove that these polynomial equations hold for all rational conformal field theories if the boundary states $|b>$ form a complete set. To this end let us recall that, by definition A^{ab}_i counts the number of boundary operators ψ^{ab}_i. These, however, are determined by a boundary algebra (Virasoro, current, or some other extended algebra)

that has the same central charges, and hence the same representations, as the bulk one. Therefore, A_i^{ab} also counts the number of different couplings $< a|\phi_i|b >$, where $|a >$ and $|b >$ denote boundary states. We have labeled the two-dimensional fields $\phi_{i,\bar{i}}$ by their chiral weights that, once all fixed-point ambiguities are resolved, determine the antichiral ones. Eq. (32) then follows if one computes the number of couplings $< a|\phi_i \ \phi_j|b >$ in two ways, by using the bulk fusion rules or by expanding in terms of a complete set of boundary states:

$$\begin{aligned} < a|\phi_i \ \phi_j|c > \quad &= \quad \sum_l N_l^{ij} \ < a|\phi_l|c > \\ &= \quad \sum_b < a|\phi_i|b >< b|\phi_j|c > = \sum_b A^{ib}{}_a \ A^j_{bc} \quad . \end{aligned} \tag{34}$$

Finally, eq. (33) follows from the general structure of the vacuum-channel annulus amplitude of eq. (22), which implies that the bilinears are totally symmetric in their boundary indices.

Eqs. (32) and (33) do not determine completely the matrices A_i^{ab}, since they contain only chiral data. Another crucial ingredient of the construction, the torus modular invariant, determines the non-vanishing disk one-point functions, and thus the range of the boundary indices, generally smaller than the range of the bulk indices. We have verified that, with this proviso, eqs. (32) and (33) have a unique solution, up to a relabeling of the boundary indices, for all the models that we have analyzed, and in particular for minimal and $SU(2)$ WZW models. For the D_{odd} series, the solution is implied by eqs. (31). In general, one can pick a subset of linearly independent A matrices, but when the boundary algebra is extended (as in the D_{odd} models) one can not interpret them as the fusion-rule coefficients of any conformal model. Some useful corollaries can be obtained even without solving eqs. (32) and (33) explicitly. For instance, for abelian fusion rules each sum on the right hand side of eq. (32) reduces to only one term, and this implies that no multiplicities larger than one are present in this case.

Matters look deceptively simpler if one introduces a graphical notation for A^i_{ab} and N^k_{ij}, where boundary indices correspond to dashed lines and bulk indices correspond to continuous lines (fig. 3). Then eqs. (32) and (33), together with the Verlinde algebra,

express the “planar duality” of all four-point amplitudes built out of these two kinds of three-point vertices.

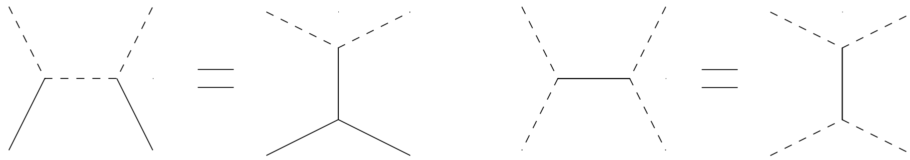

Figure 3 - Polynomial equations for A

Since the available choices of conformally invariant boundary conditions determine the possible types of (generalized) D branes, the completeness conditions are expected to play a role in the emerging picture of non-perturbative string dynamics [12, 13].

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